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# SECTION-B (ENGINEERING MATHEMATICS)

# SECTION-A
# [DISCRETE MATHEMATICS]

# CHAPTER - 1
# MATHEMATICAL LOGIC

## 1.1 INTRODUCTION
Mathematical logic is divided into two components such as
**1.** Proposition calculus
**2.** Predicate calculus

### 1. Proposition
**(i)** A proposition is a statement which is either true or false. The truth or falisty of a statement is called truth value. Since two possible truth values are admitted this logic is sometimes called two value-logic.
**(ii)** Simple statements which are represented by p, q and r are known as propositional variables and propositional variables can assume two value true or false (T and F) are called propositional constants.

> Question, exclamation and commands are not propositions.

**Example.**
**(a)** Three plus three is six 3 + 3 = 7
**(b)** The sun rises in the west. All the above statements are either true or false or these are propositions.
**(c)** Do you speak Hindi?
**(d)** What a hot day!
**(e)** 4 – x = 8 is a declarative sentence but not a statement. Since it is true or false depends on the value of x.

### 1.1.1 Compound Proposition
Simple proposition is called atomic (primary, primitive) which can't be further divided. A proposition that can be obtained from two or more propositions are called composite or compound propositions. These propositions are combined by means of logical operators called connectives.

### 1.1.2 Connectives
The words and phrases used to form compound propositions are called connectives.
The following are basic connective as shown in table.

| | Name | Symbol | Connective word | Symbolic form |
|---|---|---|---|---|
| **1.** | Negation | $\rceil$, N, $\neg$ | Not | ~ P or $\rceil$p |
| **2.** | $\wedge$ | and | Conjunction | (p) A (q) |
| **3.** | $\vee$ | OR | Disjunction | $p \vee q$ |
| **4.** | $\Rightarrow, \rightarrow$ | if …..then | Implication or condition | $p \Rightarrow q$ |
| **5.** | $\leftrightarrow$ or $\Leftrightarrow$ | If and only if | Equivalence Bicondition | $p \Leftrightarrow q$ |

# WORKBOOK

**Example 1.**
Consider following inference. If this number is divisible by 6, then it is divisible by 3.
**Solution.**
This number is not divisible by 3
p : the number is divisible by 6,
q : it is divisible by 3.
The argument is written as

$$\begin{array}{l} p \Rightarrow q \\ \underline{\ \sim q\ } \\ \therefore p \end{array}$$

Thus by modus tollens the argument is valid.

**Example 2.**
If Mr. A solved the problem, he gets answer 5.
Mr. A obtained answer as 5.
$\therefore$ Mr. A solved the problem correctly.
**Solution.**
Let p : Mr. A solved the problem
q : Mr. A obtained answer 5.

$$\begin{array}{l} p \Rightarrow q \\ \underline{q\qquad} \\ \therefore p \end{array}$$

$$\left(\left(p \Rightarrow q\right) \wedge q\right) \Rightarrow p \Rightarrow \left(\left(\overline{p} + q\right).q\right) \Rightarrow p$$

$$\Rightarrow \left(\overline{p}q + q\right) \Rightarrow p \Rightarrow q\left(\overline{p} + 1\right) \Rightarrow p$$

$q \Rightarrow \underline{p} = \overline{q} + p$ is not a tautology.

So, it is invalid

**Example 3.**
Check the validity of the following argument:-
"If the labour market is perfect then the wages of all persons in a particular employment will be equal. But it is always the case that wages for such persons are not equal therefore the labour market is not perfect."
**Solution.**
Let p: "Labour market is perfect"; q : "Wages of all persons in a particular employment will be equal". Then the given statement can be written as
[( p → q) ∧ ~ q] → ~ p
Now, [(p → q) ∧ q] ~ p = [(~ p∨q) ∧ ~ p] → p
= [(~p ∧ ~ q) ∨ (q ∧ ~ q)] → ~p
= [(~p ∧ ~q) ∨ 0] → ~ p
= [~(p ∨ q)] → ~ p
= ~[~(p ∨ q)] ∨ ~ p
= (p ∨ q) ∨ ~ p = (p ∨ ~ p) ∨ q = 1 ∨ q =1
A tautology. Hence the given statement is true.

**Example 4.**
Define tautology and contradiction. Show that "If the sky is cloudy then it will rain and it will not rain", is not a contradiction.
**Solution.**
If a compound proposition has two atomic propositions as components, then the truth table for the compound proposition contains four entries. These four entries may be all T, may be all F, may be one T and three F and so on. There are in total 16 $(2)^4$) possibilities. The possibilities when all entries in the truth table is T, implies that the compound proposition is always true. This is called tautology**.** However when all the entries are F, it implies that the proposition is never true. This situation is referred as contradiction.
Let p: "Sky is cloudy"; q : "It will rain". Then the given statement can be written as "(p → q) ∧ q".
The truth table for the expression is as below:

| **p** | **q** | **p → q** | **(p → q) ∧ ~q** |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 |

Obviously**,** (p→ q) ∧ ~ q is not a contradiction.

**Example 5.**
Consider the following open propositions over the universe $U = \{-4, -2, 0, 1, 3, 5, 6, 8, 10\}$

$P(x) : x \geq 4$

$Q(x) : x^2 = 25$

$R(x)$ : s is a multiple of 2

Find the truth values of

# ASSIGNMENT

**1.** Which of the following statement is the negation of the statement ?
"2 is even and –3 is negative"?
(a) 2 is even and –3 is not negative
(b) 2 is odd and –3 is not negative
(c) 2 is even or –3 is not negative
(d) 2 is odd or –3 is not negative

**2.** p®q is logically equivalent to
(a) $\sim p \to q$ (b) $\sim p \to q$
(c) $\sim p \wedge q$ (d) $\sim p \vee q$

**3.** Which of the following is not a well formed formula?
(a) "x [P(x)→f (x)∪ x]
(b) $\forall x_1 \forall x_2 \forall x_3 \left[\left(x_1 = x_2 \wedge x_2 = x_3\right) \Rightarrow x_1 = x_3\right]$
(c) $\sim(p \to q) \to q$
(d) $[T \vee P(a, b)] \to \exists z\, Q(z)$

**4.** $[\sim p \wedge (p \to q)] \to \sim p$ is,
(a) Satisfiable. (b) Unsatisfiable.
(c) Tautology. (d) Invalid.

**5.** The statement ( p∧q) _ p is a
(a) Contingency
(b) Absurdity
(c) Tautology
(d) None of the above

**6.** The expression $a + \bar{a}\, c$ is equivalent to
(a) $\bar{a}$ (b) $a + c$
(c) c (d) None of these

**7.** In propositional logic which one of the following is equivalent to $p \to q$
(a) $\bar{p} - q$ (b) $p - \bar{q}$
(c) $\bar{p} V q$ (d) $\bar{p} V \bar{q}$

**8.** Let p be "He is tall" and let q "He is handsome". Then the statement "It is false that he is short or handsome" is:
(a) $p \wedge q$ (b) $\sim(\sim p \vee q)$
(c) $p \vee \sim q$ (d) $\sim p \wedge q$

**9.** Which of the following proposition is a tautology?
(a) $(p \vee q) \to p$ (b) $p \vee (q \to p)$
(c) $p \vee (p \to q)$ (d) $p \to (p \to q)$

**10.** What is the converse of the following assertion?
(a) I stay if you go
(b) If you do not go then I do not stay
(c) If I stay then you go
(d) If you do not stay then you go

**11.** Which of the following statement is the negation of the statement "4 is even or –5 is negative"?
(a) 4 is odd and –5 is not negative
(b) 4 is even or –5 is not negative
(c) 4 is odd or –5 is not negative
(d) 4 is even and –5 is not negative

**12.** Which one is the contrapositive of q ® p?
(a) $p \to q$
(b) $\neg p \to \neg q$
(c) $\neg q \to \neg p$
(d) None of the se

**13.** If P and Q are propositions then $((P \vee Q) \vee \sim P)$
(a) Is a tautology (b) Is a contradiction
(c) Is a contingency (d) Is equivalent to Q

**14.** The prepositional function$(\sim(P \vee Q) \vee (\sim P \wedge Q) \vee P)$ is
(a) A tautology (b) A contradiction
(c) A contingency (d) $\Leftrightarrow P$

**15.** $\sim(P \vee (\sim P \wedge Q)) \Leftrightarrow$ (is equivalent to )
(a) $P \vee Q$ (b) $P \wedge Q$
(c) $\sim P \vee \sim Q$ (d) $\sim P \wedge \sim Q$

**16.** The simplest form of $((P \to Q) \rightleftarrows (\sim P \vee Q) \wedge R$ is
(a) T (b) F
(c) R (d) $P \to R$

# CHAPTER - 2
## GRAPH THEORY

### 2.1 INTRODUCTION
**1.** A graph G is a triple consisting of a vertex set V(G), an edge set E(G), and a relation that associates with each edge between two vertices called its end points. Vertices are sometimes called as nodes.
**2.** A loop is an edge whose end points are equal. Multiple edges are edges having the same pair of end points.

#### 2.1.1 Types of Graphs
There are following types of graphs as
**1.** Simple Graph
**2.** Multigraph
**3.** Psuedograph
**4.** Undirected Graph
**5.** Digraph or Directed Graph

***1. Simple Graph***
A simple graph is an undirected graph having no loops or multiple edges. We specify a simple graph by its vertex set and edge set and edge set treating the edge set as a set of unordered pairs of vertices (undirected graph) and writing e = {u, v} for an edge e with endpoints u and v.

When u and v are endpoints of an edge, they are adjacent and are neighbors. We write u $\leftrightarrow$ v for u is adjacent to v and we say that the edge e = {u, v} is incident on u and v.

***2. Multigraph***
A Multigraph is an undirected graph in which multiple edges between pairs of vertices allowed. However, self loops are not allowed.

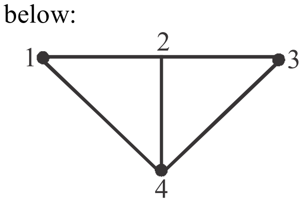
***3. Psuedograph***
A psuedograph is an undirected graph in which multiple edges as well as self loops are allowed.

***4. Undirected Graph***
A graph in which the edges do not have direction. i.e. A graph (V, E) such that E is a set of undirected edges that are unordered pairs of vertices of V.

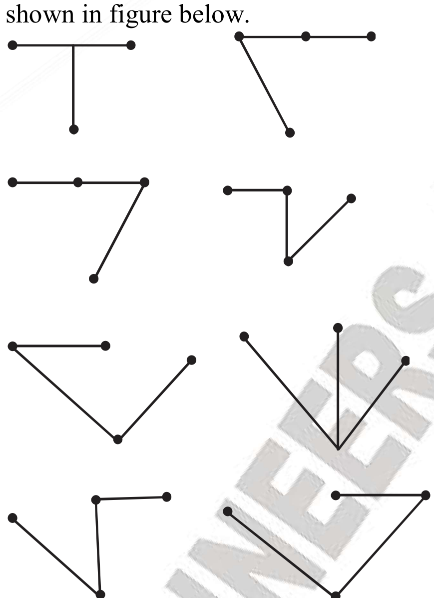
***5. Digraph or Directed Graph***
A graph in which the edges have direction. i.e. A graph (V, E) such that E is a set of directed edges that are ordered pairs of vertices of V.

# WORKBOOK

**Example 1.** Find all spanning trees of the graph G shown below:

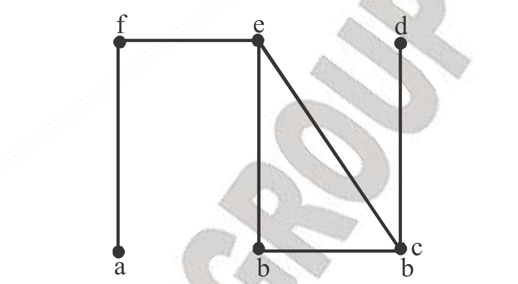



**Solution.**
The graph G has four vertices and hence each spanning tree must have 4 – 1 = 3 edges. Thus each tree can be obtained by deleting two of the five edges of G. This can be done in 10 ways, except that two of the ways lead to disconnected graphs. Thus there are eight spanning trees as shown in figure below.

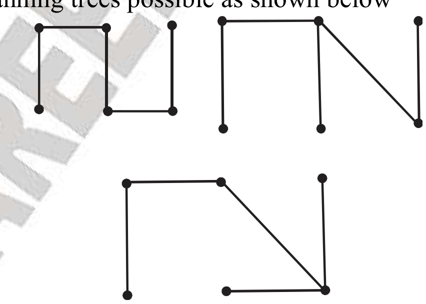


Alternatively Kirchoff's method using matrices may be used to find the number of spanning trees of a graph.

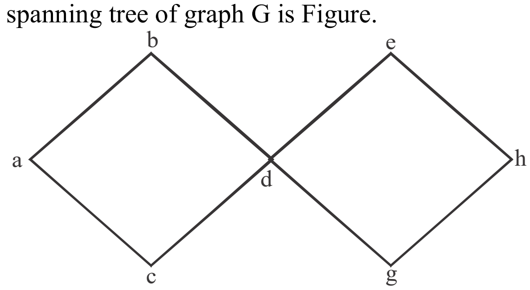
**Example 2.** Find all spanning trees for the graph G shown in figure by removing edges in simple circuits.




**Solution.**
The graph G has one cycle cbec and removal of any edge of the cycle gives a tree. There are three edges in the cycle and hence there are 3 spanning trees possible as shown below



**Example 3.** Use BFS algorithm to find a spanning tree of graph G is Figure.




**Solution.**
**(i)** Choose the vertex a to be the root.
**(ii)** Add edges incident with all vertices adjacent to a so that edges {a, b}, {a, c} are added. Two vertices b and c are in level 1 in the tree.
**(iii)** Add edges from these vertices at level 1 to adjacent vertices not already in the tree. Hence the edge{c, d} is added. The vertex d is in level 2. (Now, {b, d} is not joined since it will form a cycle)

# CHAPTER - 3
# *RELATION*

### 3.1 INTRODUCTION
The most direct way of expressing Relationship between elements of two set is use ordered pair made up of two related elements.

**Example.**
A is (Apple, Carrot, Milk)
B is (Mango, Water, Radish)
There is a Relation e is a same category, between two sets A and B.
R is {(Apple, Mango) (Carrot, Radish) (Milk, Water)}= $\{(x, y): x \in A, y \in B, xRy\}$
Thus, relation is same category of from set A to B that gives a subset R= $A \times B$ such that $(x, y) \in R$ if and only if x R y

### 3.1.1 Definition
Let A and B be two sets , a relation from A to B is a subset of Cartesian product $A \times B$. Suppose R is relation from A to B. Then R is set ordered pair (a, b) where $a \in A$ and $b \in B$. Every such pair is a R b and read as a relatable to b. R is binary relation from A to B since the elements of set R are ordered pairs. If we use term relation on its own, then binary Relation is implied.

$R \subseteq A \times B$ if $(x, y) \in R$
1. $R \subseteq A \times B$ given by $R = \{(x, y) : x \in A, y \in B\}$
2. A Relation R on set A is a subset of $A \times A$ and is called binary relation

***1. Domain (R)***
It is the set of first elements of ordered pair. It is formally expressed as $\{x : (x, y) \in R\}$
***2. Range (R)***
It is the set of first elements of ordered pair. If is formally expressed as $\{y : (x, y) \in R\}$

Let the number of elements of A and B are m and n respectively . No of elements in AB is mn. Therefore number of elements in power set of $A \times B$ has $2^{mn}$. Thus $A \times B$ has $2^{mn}$ different Subsets. Now every subset of $A \times B$ is relation from A to B. Hence the number of Different relation from A to B is $2^{mn}$

### 3.1.2 Types of Relations
There are following types of relations
**1.** Inverse Relation
**2.** Identity Relation
**3.** n-ary Relation

# GATE QUESTIONS

**1.** A binary relation R on N×N is defined as follows: (a,b)R(c,d) if a≤c or b≤d. Consider the following
propositions:
P: R is reflexive
Q: R is transitive
Which one of the following statements is **TRUE**?

**[GATE - 2016]**

(a) Both P and Q are true.
(b) P is true and Q is false.
(c) P is false and Q is true.
(d) Both P and Q are false.

**2.** Let R be the relation on the set of positive integers such that aRb if and only if a and b are distinct and have a common divisor other than 1. Which one of the following statements about R is true?

**[GATE - 2015]**

(a) R is symmetric and reflexive but not transitive
(b) R is reflexive but not symmetric and not transitive
(c) R is transitive but not reflexive and not symmetric
(d) R is symmetric but not reflexive and not transitive

**3.** The cardinality of the power set of {0, 1, 2, …, 10} is ________

**[GATE - 2015]**

**4.** Suppose U is the power set of the set S = {1,2,3,4,5,6}. For any T ∈ U, let |T| denote the number of elements in T and T ′ denote the complement of T. For any T, R ∈ U, let T \ R be the set of all elements in T which are not in R. Which one of the following is true?

**[GATE - 2015]**

(a) $\forall X \in U(|X| = |X'|)$
(b) $\exists X \in U\ \exists Y \in U(|X| = 5, |Y| = 5 \text{ and } X \cap Y = \varnothing)$
(c) $\forall X \in U\ \forall Y \in U(|X| = 2, |Y| = 3 \text{ and } X \setminus Y = \varnothing)$
(d) $\forall X \in U\ \forall Y \in U(X \setminus Y = Y' \setminus X')$

**5.** Consider the following relation on subsets of the set S of integers between 1 and 2014. For two distinct subsets U and V of S we say U < V if the minimum element in the symmetric difference of the two sets is in U.
Consider the following two statements:
S1: There is a subset of S that is larger than every other subset.
S2: There is a subset of S that is smaller than every other subset.
Which one of the following is CORRECT?

**[GATE - 2014]**

(a) Both S1 and S2 are true
(b) S1 is true and S2 is false
(c) S2 is true and S1 is false
(d) Neither S1 nor S2 is true

**6.** A pennant is a sequence of numbers, each number being 1 or 2. An n-permant is a sequence of numbers with sum equal to n. For example, (1, 1, 2) is a 4-pennant. The set of all possible 1-pennants is {(1)}, the set of all possible 2-pennants is {(2), (1, 1)} and the set of all 3-pennants is {(2, 1), (1, 1, 1), (1, 2)}. Note that the pennant (1, 2) is not the same as the pennant (2, 1). The number of 10-pennants is _______.

**[GATE - 2014]**

**7.** What is the possible number of reflexive relations on a set of 5 elements?

**[GATE - 2010]**

(a) $2^{10}$ (b) $2^{15}$
(c) $2^{20}$ (d) $2^{25}$

**8.** Consider the binary relation R = {(x, y), (x, z), (z, x), (z, y)} on the set {x, y, z}. which one of the following is TRUE?

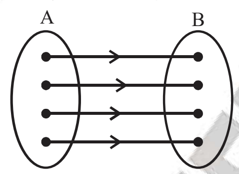
**[GATE - 2009]**

(a) R is symmetric but NOT antisymmetric

# CHAPTER - 4
## *FUNCTIONS*

### 4.1 INTRODUCTION
A function is special case of relation. Let A and B be two non-empty sets and R be a relation from A to B, then R may not relate an element of A to an element of B or it may relate an element of A to more than one element of B. But a function relates each element of A to a unique element of B.

**1. Definition**
Let A and B two non-empty sets. A function f from A to B is set of ordered pairs.
$f \subseteq A \times B$ with a property that for each element x in A there is unique element y in B such that (x, y) $\in$ f. It is Represented by $f : A \rightarrow B$ or $A \xrightarrow{F} B$




**(i)** There may be some elements of set B which are not associated to any element of set A.
**(ii)** That each element of set A must be associated to one and only one element of B.
If f is a function from A to B, then A is called domain of f and the set B is called co-domain.
**(iii)** If (x, y) $\in$ f, we can write it y = f(x) y is image of x and x is called pre-image of y. The set consisting of all images of the elements of A under the function f is called range of f. It is denoted by f(A). The range of f={(x):for all x $\in$ A}

| |
|---|
| Range is subset of codomain which may or may not be equal to B. |

**Example.**
Let A = {1, 2, 3, 4, 5} B ={0, 1, 2, 3, 5, 7, 9, 12, 13} and f = (1 1), (2, 0) (3, 7), (4, 9) (5, 12) then f is a function from A to B because each elements of A has a unique image in B and no element of A has two or more images in B. Range of $f = \{1, 0, 7, 9, 12\}$

### 4.1.1 Types of Function
A function can be any of the following types as
**1.** One to one
**2.** Many to one
**3.** Into
**4.** Onto
**5.** Bijective
**6.** Identity
**7.** Composition of function
**8.** Inverse
**9.** Symmetric

# CHAPTER - 5
## GROUP THEORY & LATTICES

### 5.1 BINARY OPERATION
**1.** Let G be a non empty set. Then $G \times G = \{(a, b) : a \in G, b \in G\}$
If $f : G \times G \to G$ then f is said to binary operation on G. Binary operation on G is a function that assigns each ordered pair of element of G an element of G.

**2.** The symbol +, ·, 0, ∗ etc are used to denote binary operation on a set. Thus + will be a binary operation on G if and only if. $a + b \in G$ for all $a, b \in G$ and $a + b$ is unique. This is called closure property.

**3.** + and × are closed when $a + b \in N$ for all $a, b \in N$ and $a \times b \in N$ for all $a, b \in N$

**4.** A binary operation is sometimes called a composition in G.

**5.** For finite set, a binary operation on the set can be defined by means of a Table called composite table.

**Example.**
Let S = {a, b, c}. Following table defines ∗ on S as

| ∗ | a | b | c |
|---|---|---|---|
| a | c | b | a |
| b | a | a | a |
| c | b | b | b |

To, Determine the elements S assigned to a ∗ b, we look at the intersection of row labeled by a and the element headed by b then a ∗ b = b and b ∗ a = a.

### 5.1.1 Algebraic Structure
A non-empty set together with one or more than one binary operation is called algebraic structure. (N, +), (z, +) (R, +, ·) are all algebraic structures.

### 5.1.2 Properties of Binary Operations
***1. Closure***
It states that if elements $a, b \in A$ and $a * b \in A$ then it is said that set A is closed under operation ∗

***2. Associativity***
A binary operation ∗ on a set is said to be associative if and only if for any element $a, b, c \in S$ $a * (b * c) = (a * b) * c$

***3. Commutative law***
a and $b \in S$, then $a * b = b * a$

**Example.**
Algebraic structure (z, +) (z, ·) where binary operation of addition and multiplication on Z both are associative and commutative since addition and multiplication of integers is both associative and commutative.

# WORKBOOK

**Example 1.** N = {set of positive integers} $*$ is operation of LCM on N. Is it a Semi-Group, Is it commutative.?

**Solution.**

$a * b = LCM (a, b)$

$(a * b)*c = (LCM (a, b)) * c$

$= LCM (LCM (a, b), e]$

$= LCM [a, LCM(b, c)] = a * (b * c)$

Hence $*$ is associative

It is also closed, so it is a Semi-group

It is also commutative

$\therefore$ LCM (a, b) = LCM (b, a) = a $*$ b.

**Example 2.** Show that root unity z = {1, w, $w^2$) and multiplication operator is Abelian group.

**Solution.**

Since $w^3 = 1$, so we can prepare a composite table.

| **X** | **1** | **w** | **$w^2$** |
|---|---|---|---|
| 1 | 1 | w | $w^2$ |
| w | w | $w^2$ | 1 |
| $w^2$ | $w^2$ | 1 | w |

From this

**(1)** Closure property: Since the table consists of elements which are the element of algebraic structure so closed.

**(2)** Associative law: Multiplication is associative on complex number since z are complex numbers.

**(3)** Commutative law

$1 \times w = w \times 1$

$w \times w^2 = w^2 \times w$

**(4)** identity element a $*$ e = a so $1 \times 1 = 1$ $w \times 1 = w$ $w^2 \times 1 = w^2$, so 1 is identity

**(5)** Inverse clearly $i^{-1} = 1$; $w^{-1} = w^2$ $(w^2)^{-1} = w$ so it is an Abelian group.

**Example 3.** Show that the set {1, 2, 3, 4, 5} is not group under addition modulo 6.

**Solution.**

Since G = {1, 2, 3, 4, 5}

Hence composition table

| $t_6$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 3 | 4 | 5 | 0 | 1 |
| 3 | 4 | 5 | 0 | 1 | 2 |
| 4 | 5 | 0 | 1 | 2 | 3 |
| 5 | 0 | 1 | 2 | 3 | 4 |

Since all entries in composition table do not belong to G, in particular $0 \notin G$. Hence it is not closed.So neither Semigroup, nor monoid nor group.

**Example 4.** Show that H = {3n : n $\in$ z} is a subgroup of (z, +)

**Solution.**

H is non empty and we know that necessary & sufficient condition for subgroup is a$\in$H, b$\in$H $\Rightarrow a * b^{-1} \in H$

Such that x = 3p, y = 3q

Let x, y $\in$ H $\therefore$ p, q exists $\in$z

Now:- $xy^{-1} = 3p - 3q = 3(p - q)$

Thus $xy^{-1} \in H$ Hence H is subgroup.

**Example 5.** The group (G, $+_6$) is a cyclic group, **Solution.**

G = {0, 1, 2, 3, 4, 5}.

Sol. $1^1 = 1$, $1^2 = 1 +_6 1^3 = 1 + 1 \bmod 6 = 2$

$1 +_6 1^2 = 3$, $1 +_6 1^3 = 4$, $1 +_6 1^4 = 5$

$1 + 1^5 = 0$

Thus G = {$1^0, 1^1, 1^2, 1^3, 1^4, 1^5, 1^6 = 0$}

Where G is cyclic group & 1 is generator.

**Example 6.** If f is a homomorphism from a commutative semigroup (S, $*$) onto a semigroup (T, $*'$), then show that (T, $*'$) is also commutative.

**Solution.**

In any distributive lattice L, for any three elements a, x, y of L, we can write

$x = x \vee (a \wedge x)$

$= (x \vee a) \wedge (x \vee x)$ [Distribute law]

$= (a \vee y) \wedge x$ [Commutative and

# SECTION-B
# [ENGINEERING MATHEMATICS]

# CHAPTER - 1
# *LINEAR ALGEBRA*

## 1.1 INTRODUCTION

Linear Algebra and matrix theory occupy an important place in modern mathematics and has applications in almost all branches of engineering and physical sciences. An elementary application of linear algebra is to the solution of a system of linear equations in several unknowns, which often result when linear mathematical models are constructed to represent physical problems. Nonlinear models can often be approximated by linear ones. Other applications can be found in computer graphics and in numerical methods.

In this chapter, we shall discuss matrix algebra and its use in solving linear system of algebraic equations $A\hat{x} = b$ and in solving the eigen value problem $A\hat{x} = \lambda\hat{x}$.

## 1.2 ALGEBRA OF MATRICES

### 1.2.1 Matrix Definition

A system of mn numbers arranged in the form of a rectangular array having m rows and n columns is called an matrix of order m × n.

If $A = [a_{ij}]_{m\times n}$ be any matrix of order m × n then it is written in the form:

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12}..........a_{1n} \\ a_{21} & a_{22}..........a_{2n} \\ .... & .................. \\ .... & .................. \\ a_{m1} & a_{m2}..........a_{mn} \end{bmatrix}$$

Horizontal lines are called rows and vertical lines are called columns.

### 1.2.2 Types of Matrices

***1. Square Matrix***

An m × n matrix tor which m = n (The number of rows is equal to number of columns) is called square matrix. It is also called an n-rowed square matrix. i.e. The elements $a_{ij}$ + I =j, i.e. $a_{11}$, $a_{22}$ .... are called **DIAGONAL ELEMENTS** and the line along which they lie is called **PRINCIPLE DIAGONAL** of matrix. Elements other than $a_{11}$, $a_{22}$, etc are called off-diagonal elements i.e. $a_{ij}| \neq j$.

**Example.** $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 9 & 8 & 3 \end{bmatrix}_{3\times 3}$ is a square Matrix

A square sub matrix of a square matrix A is called a **"principle sub-matrix"** it its diagonal elements are also the diagonal elements of the matrix A.



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# ASSIGNMENT

**1.** The rank of the matrix $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 3 & 6 & 9 & 12 \\ 0 & 6 & 10 & 15 & 20 \end{bmatrix}$ is

(a) Zero (b) 1
(c) 2 (d) 3

**2.** A square matrix a is invertible if and only if
(a) It has non zero element
(b) Determinant of A is zero
(c) Determinant of A is non zero
(d) Has all elements not equal to zero

**3.** If A is a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

(a) $A(\text{Adj } A) = |A|I$ (b) $|A^{-1}| = (|A|)^{-1}$
(c) $|\text{adj } A^{-1}| = |A|$ (d) $|\text{adj } A| = |A^{-1}|$

**4.** If A $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$ B = $\begin{bmatrix} 1 & 2 \\ -2 & 3 \\ 3 & 1 \end{bmatrix}$ C= $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Are matrices, then the order of $(5A - 3B)C$ is
(a) $5 \times 1$
(b) $2 \times 1$
(c) $3 \times 1$
(d) Matrix does not exist

**5.** The matrix $\begin{bmatrix} 0 & 3 & 5+2i \\ -3 & 0 & -9 \\ -5 & 9 & 0 \end{bmatrix}$

(a) Symmetric matrix
(b) Skew-symmetric matrix
(c) Hermitian matrix
(d) skew-Hermitian matrix

**6.** Let A be square matrix and $A^t$ be its transpose matrix then $A - A^t$ is
(a) Symmetric matrix
(b) Skew-symmetric matrix
(c) Zero matrix
(d) Identity matrix

**7.** The rank of the matrix: $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ is

(a) 1 (b) 2
(c) 3 (d) 4

**8.** The system of linear equation.
$$x + 2y + 3z = \lambda x$$
$$3x + y + 2z = \lambda y$$
$$2s + 3y + z = \lambda z$$
has a non-zero solution when $\lambda$ equals
(a) 2 (b) 4
(c) 6 (d) 8

**9.** If $A = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$ then $A^3 + A = 0$ whenever

(a) $\alpha\beta = 0$ (b) $\alpha\beta = 1$
(c) $\alpha\beta \neq 0$ (d) $\alpha\beta = -1$

**10.** If $A = \begin{pmatrix} 1 & 0 & -1 \\ -2 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ then inverse of matrix A will be :

(a) $\begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -2 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 0 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix}$

**11.** Consider the equation AX = B where

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# GATE QUESTIONS

**1.** Consider a matrix P whose only eigenvectors are the multiples of $\begin{bmatrix}1\\4\end{bmatrix}$.

Consider the following statements:
(i)P does not have an inverse.
(ii)P has a repeated eigenvalue.
(iii)P cannot be diagonalized.
Which one of the following options is correct ?

**(GATE - 2018)**

(a)Only i and iii are necessarily true
(b)Only ii is necessarily true
(c)Only i and ii are necessarily true
(d)Only ii and iii are necessarily true

**2.** Consider a matrix $A = uv^T$ where $u = \begin{pmatrix}1\\2\end{pmatrix}, v = \begin{pmatrix}1\\1\end{pmatrix}$. Note that $v^T$ denotes the transpose of v. The largest eigenvalue of A is ________.

**(GATE - 2018)**

**3.** Let A be n × n real valued square symmetric matrix of rank 2 with $\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}^2 = 50$. Consider the following statements.
(I) One eigenvalue must be in [–5, 5]
(II) The eigenvalue with the largest magnitude must be strictly greater than 5
Which of the above statements about engenvalues of A is/are necessarily Correct?

**[GATE - 2017]**

(a) Both (I) and (II)
(b) (I) only
(c) (II) only
(d) Neither (I) nor (II)

**4.** Let $c_1, \ldots, c_n$ be scalars, not all zero, such that $\sum_{i=1}^{n} c_i a_i = 0$ where $a_i$ are column vectors in $R''$. Consider the set of linear equations
$Ax = b$
Where A is $[a_1, \ldots, a_n]$ and $b = \sum_{i=1}^{n} a_i$. The set of equations has

**[GATE - 2017]**

(a) A unique solution at $x = J_n$ where $J_n$ denotes a n-dimensional vector of all 1
(b) No solution
(c) Infinitely many solutions
(d) Finitely many solutions

**5.** If the characteristic polymnomial of a 3 × 3 matrix m over R(the set of real numbers) is $\lambda^3 - 4\lambda^2 + a\lambda + 30$. $A \in R$, and one eigenvalue of M is 2, then the largest among the absolute values of the eigen values of m is _______.

**[GATE - 2017]**

**6.** Let $P = \begin{bmatrix}1 & 1 & -1\\ 2 & -3 & 4\\ 3 & -2 & 3\end{bmatrix}$ and

$Q = \begin{bmatrix}-1 & -2 & -1\\ 6 & 12 & 6\\ 5 & 10 & 5\end{bmatrix}$ be two matrices.

Then the rank of P + Q is _______.

**[GATE - 2017]**

**7.** The rank of the matrix

$\begin{bmatrix}1 & -1 & 0 & 0 & 0\\ 0 & 0 & 1 & -1 & 0\\ 0 & 1 & -1 & 0 & 0\\ -1 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & -1\end{bmatrix}$ is _______.

**[GATE - 2017]**

**8.** The eigen values of the matrix given below are $\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -3 & -4\end{bmatrix}$

**[GATE - 2017]**

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# CHAPTER - 2
# *CALCULUS*

## 2.1 LIMIT
### 2.1.1 Definition
A number 00001 A is said to be limit of function f (×) at × = a if for any arbitrarily chosen positive integer ∈, however small but not zero there exist a corresponding number δ greater than zero such that: $|f(x) - A| < \in$ or all values of × for which 0 < |× − a| < ∂ where | × − a| means the absolute value of (× – a) without any regard to sign.

### 2.1.2 Right and Left Hand Limits
If × approaches a from the right, that is, from larger value of × than a, the limit of f as defined before is called the right hand limit of f(×) and is written as:

$\underset{x\to a+0}{\text{Lt}} f(x)$ or f(a+0) or $\underset{x\to a^{+}}{\text{Lt}} f(x)$

Working rule for finding right hand limit is, put a + h for × in f(×) and make h approach zero.

In short, we have, $f(a+0) = \lim\limits_{h\to 0^{f(a+h)}}$

Similarly if × approaches a from left, that is from smaller values of × than a, the limit of f is called the left hand limit and is written as:

$\underset{x\to a-0}{\text{Lt}} f(x)$ or f(a-0) or $\underset{x\to a^{-}}{\text{Lt}} f(x)$

In this case, we have f(a–0) = $\lim\limits_{h\to 0^{f(a-h)}}$

In both right hand and left hand limit of f, as $x \to a$ exist and are equal in value, their common value, evidently, will be the limit of f as $x \to a$. If however, either or both of these limits do not exist, the limit of f as $x \to a$does not exist. Even if both these limits exist but are not equal in value then also the limit of f as $x \to a$ does not exist.

∴ when $\underset{x\to a}{\text{Lt}}$ f(x) = $\underset{x\to a^{-}}{\text{Lt}} f(x)$

then $\underset{x\to a}{\text{Lt}} f(x) = \underset{x\to a^{+}}{\text{Lt}} f(x) = \underset{x\to a^{-}}{\text{Lt}} f(x)$

Limit of a function can be any real number, $\infty$ or $-\infty$. It can sometimes be $\infty$ or □ $\infty$, which are also allowed values for limit of a function.

**Various Formulae**
These formulae are sometimes useful while taking limits.

**1.** $(1+x)^{n} = 1 - nx + \frac{n(n-1)}{2!}x^{2} + \frac{n(n-1)(n-2)}{3!}x^{3} + ...$

**2.** $(1-x)^{-1} = 1 + x + x^{2} + x^{3} + ....$

**3.** $a^{x} = 1 + \times \log a + \frac{x^{2}}{2!}(x\log a)^{2} + \frac{x^{3}}{3!}(x\log a)^{3} + ...$

# WORKBOOK

**Example 1.** What is the value of $\lim\limits_{x\to 0}\dfrac{\sin\left[\frac{4}{3}x\right]}{x}$?

**Solution.**
We have

$$\lim_{x\to 0}\frac{\sin\left[\frac{4}{3}x\right]}{x} = \lim_{\frac{4}{3}x\to 0}\frac{4}{3}\frac{\sin\left[\frac{4}{3}x\right]}{\frac{4}{3}x}$$

$$=\frac{4}{3}\lim_{\frac{4}{3}x\to 0}\frac{\sin\left[\frac{4}{3}x\right]}{\frac{4}{3}x}=\frac{4}{3}\times 1=\frac{4}{3}$$

**Example 2.** What is the value of
$\lim\limits_{x\to 0}\dfrac{x^3-6x^2+11x-6}{x^2-6x+8}$?

**Solution.**
When $x\to 2, \dfrac{x^3-6x^2+11x-6}{x^2-6x+8}=\dfrac{0}{0}$

Hence, we apply L'Hospital's rule,

$$\lim_{x\to 2}\frac{3x^2-12x+11}{2x-6}=\frac{3(2)^2-12(2)+11}{2(2)-6}$$

$$=\frac{12-24+111}{-2}=\frac{-1}{-2}=\frac{1}{2}$$

**Example 3.** If a function is given by

$$f(x)=\begin{cases}\dfrac{\sin x}{x}+\cos x & x\neq 0\\ 2, & x=0\end{cases}$$

Find out whether or not f(x) is continuous at x = 0.

**Solution.**
We have
L.H.L at x = 0

$$=\lim_{x\to 0}f(x)=\lim_{h\to 0}f(o-h)=\lim_{h\to 0}f(-h)$$

$$=\lim_{h\to 0}\left[\frac{\sin(-h)}{-h}+\cos(-h)\right]=1+1=2$$

R.H.L. at x = 0

$$=\lim_{x\to 0}f(x)=\lim_{h\to 0}f(o+h)=\lim_{h\to 0}f(h)$$

$$=\lim_{h\to 0}\left[\frac{\sin h}{h}+\cos h\right]=1+1=2$$

Also, we k now that f(0) = 2.
Thus, $\lim\limits_{h\to 0^-}f(x)=\lim\limits_{h\to 0^+}f(x)=f(0)$.
Hence, f(x) is continuous at x = 0.

**Example 4.** Discuss the continuity of the function f(x) at x = ½, where

$$f(x)=\begin{cases}1/2^{-x}, & x\le x<1/2\\ 1, & x=1/2\\ 3/2^{-x}, & 1/2<x\le 1\end{cases}$$

**Solution.**
We have

L.H.L. at $x=\dfrac{1}{2}$

$$=\lim_{x\to 1/2^-}f(x)\lim_{x\to 1/2}\left(\frac{1}{2}-x\right)=\frac{1}{2}-\frac{1}{2}=0$$

R.H.L. $x=\dfrac{1}{2}$

$$=\lim_{x\to 1/2^+}f(x)\lim_{x\to 1/2}\left(\frac{3}{2}-x\right)=\frac{3}{2}-\frac{3}{2}=1$$

Since, $\lim\limits_{x\to 1/2^-}f(x)\neq\lim\limits_{x\to 1/2^+}f(x)$

Hence, f(x) not continuous at $x=\dfrac{1}{2}$.

**Example 5.** Discuss the continuity of f(x) = 2x – |x| at x = 0.

**Solution.**
We have

$$f(x)=2x-|x|=\begin{cases}2x-x, & \text{if } x\ge 0\\ 2x-(-x), & \text{if } x<0\end{cases}$$

$$\Rightarrow f(x)=\begin{cases}x, & \text{if } x\ge 0\\ 3x, & \text{if } x<0\end{cases}$$

Now,
L.H.L. at x = 0

# ASSIGNMENT

**1.** $\lim_{x\to 0} x \log_x$ equals
(a) 1 (b) 0
(c) 1/2 (d) 1/3

**2.** If x = r cos θ, y = r sin θ; then the value of $\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}$ is
(a) 0 (b) 1
(c) $\frac{\partial r}{\partial x}$ (d) $\frac{\partial x}{\partial y}$

**3.** $\lim_{n\to\infty}\frac{I^2+2^n+....+n^2}{n^3}$ equals.
(a) 1/4 (b) 1/2
(c) 1/6 (d) 1/3

**4.** The value of the integral $\iint xy\,dx\,dy$. Taken over the region bounded by the two axes and the straight line x + y = 1.
(a) 1/20 (b) 1/24
(c) 1/30 (d) 1/40

**5.** For the function f(x) = |x| language's mean value theorem does not hold in the interval
a) [−1,0] (b) [0,1/2]
(c) [0,1] (d) [−1,1]

**6.** The value of $\int_0^1\int_0^1 (x^2+y^2)dx\ dy$ is
(a) 1 (b) 0
(c) 1/3 (d) 2/3

**7.** The point of inflexion of curve y = $x^{5/2}$ is
(a) (1,1) (b) (0,0)
(c) (1,0) (d) (0,1)

**8.** The value of
$$\lim_{n\to\infty}\left[\frac{n^{1/2}}{n^{3/2}}+\frac{n^{1/2}}{(n+3)^{3/2}}+.....\frac{n^{1/2}}{(n+3)(n-1)^{3/2}}\right]$$
(a) $\int_0^1\frac{dx}{(1+3x)^{3/2}}$ (b) $\int_0^\infty\frac{dx}{(1+3x)^{3/2}}$
(c) $\int_0^1\frac{dx}{(1+3x)^{3/1}}$ (d) None

**9.** If u = log($x^3$ + $y^3$ + $z^3$ – 3xyz) then the value of $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$ is
(a) $\frac{3}{(x+y+z)^3}$ (b) $\frac{-9}{(x+y+z)^2}$
(c) $\frac{9}{(x+y+z)}$ (d) $\frac{3}{(x+y+z)^2}$

**10.** The value of $\int_0^{\pi/2}\frac{(\cos x-\sin x)dx}{1+\sin x\cos x}$
(a) 1 (b) 1/2
(c) 0 (d) 2

**11.** Let $f(x)=\begin{cases} x\sin\frac{1}{x} \text{ if } x\neq 0 \\ \text{if } x=0\end{cases}$. Then at $x=0$, f is
(a) Continuous but not differentiable
(b) Not continuous
(c) Differentiable
(d) Neither continuous nor differentiable

**12.** The function f(x,y) may have a maxima or minima at a point if at that point -
(a) $\left[\frac{\partial^2 f}{dx^2}\cdot\frac{\partial^2 f}{\partial y}-\left(\frac{\partial^2 f}{\partial xdy}\right)^2\right]>0$
(b) $\left[\frac{\partial^2 f}{dx^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial xdy}\right)^2\right]<0$
(c) $\left[\frac{\partial^2 f}{dx^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial xdy}\right)^2\right]=0$
(d) None of these

# GATE QUESTIONS

**1.** The value of $\int_{0}^{\pi/4} x\cos\left(x^2\right)dx$ correct to three decimal places (assuming that π=3.14) is ________

**(GATE - 2018)**

**2.** The value of $\lim_{x\to 1}\frac{x^7-2x^5+1}{x^3-3x^2+2}$

**[GATE - 2017]**

(a) is 0 (b) is –1
(c) is 1 (d) Does not exit

**3.** If $f(x)=R\sin\left(\frac{\pi x}{2}\right)+S.f'\left(\frac{1}{2}\right)=\sqrt{2}$ and $\int_0^1 f(x)dx=\frac{2R}{\pi}$, then the constants R and S are respectively.

**[GATE - 2017]**

(a) $\frac{2}{\pi}$ and $\frac{16}{\pi}$ (b) $\frac{2}{\pi}$ and 0
(c) $\frac{4}{\pi}$ and 0 (d) $\frac{4}{\pi}$ and $\frac{16}{\pi}$

**4.** An integral I over a counter clock wise circle C is given by

$$I=\oint_C \frac{z^2-1}{z^2+1}e^z dz$$

If C is defined as |z| = 3, then the value of I is

**[GATE - 2017]**

(a) –πi sin(1) (b) –2πi sin (1)
(c) –3πi sin (1) (d) –4πi sin (1)

**5.** The minimum value of the function $f(x)=\frac{1}{3}x(x^2-3)$ in the interval $-100 \le x \le 100$ occurs at x = ____

**[GATE - 2017]**

**6.** The value of the contour integral in the complex – plane $\oint\frac{z^3-2z+3}{z-2}dz$ along the contour |z| = 3, taken counter - clockwise is

**[GATE - 2017]**

(a) –18πi (b) 0
(c) 14πi (d) 48πi

**7.** Let $g(x)=\begin{cases}-x, & x\le 1\\ x+1 & x\ge 1\end{cases}$ and

$$f(x)=\begin{cases}1-x, & x\le 0\\ x^2 & x>0\end{cases}$$

Consider the composition of f and g, i.e., (f o g)(x) = f(g(x)). The number of discontinuities in (f o g)(x) present in the interval (–∞, 0) is

**[GATE - 2017]**

(a) 0 (b) 1
(c) 2 (d) 4

**8.** Let $y^2 - 2y + 1 = x$ and $\sqrt{x}+y=5$. The value of $x+\sqrt{y}$ equals _____. (Given the answer up to three decimal places)

**[GATE - 2017]**

**9.** A function f(x) is defined as $f(x)=\begin{cases}e^x, x<1\\ \ln x+ax^2+bx, x\ge 1\end{cases}$, where $x \in R$.
Which one of the following statements is TRUE?

**[GATE - 2017]**

(a) f(x) is NOT differentiable at x = 1 for any values of a and b.
(b) f(x) is differentiable at x = 1 for the unique values of a and b.
(c) f(x) is differentiable at x = 1 for all values of a and b such that a + b = e.
(d) f(x) is differentiable at x = 1 for all values of a and b.

# CHAPTER - 3
## *PROBABILITY AND STATICS*

### 4.1 PROBABILITY FUNDAMENTALS
#### 4.1.1 Definitions

Sample Space and Event: Consider an experiment whose outcome is not predictable with certainty. Such an experiment is called a random experiment. However, although the outcome of the experiment will not be known in advance, let us suppose that the set of all possible outcomes is known. This set of all possible outcomes of an experiment is known as the sample space of experiment and is denoted by S. Some examples follow.

**1.** If the outcome of an experiment consist in the determination of the sex of a newborn child, then S = {g, b} where the outcome g means that the child is a girl and b is the boy.

**2.** If the outcome of an experiment consist of what comes up on a single dice, then S = {1, 2, 3, 4, 5, 6}.

**3.** If the outcome of an experiment is the order of finish in a race among the 7 horses having post positions 1, 2, 3, 4, 5, 6, 7; then S = {all 7! permutations of the (1, 2, 3, 4, 5, 6, 7)}.

The outcome (2, 3, 1, 6, 5, 4, 7) means, for instances, that the number 2 horse comes in first, then the number 3 horse, then the number 1 horse, and so on.

Any subset E of the sample space is known as Event. That is, an event is a set consisting of some or all of the possible outcomes of the experiment. For example, in the throw of a single dice S = {1, 2, 3, 4, 5, 6} and some possible events are

$E_1 = \{1, 2, 3\}$ $\quad$ $E_2 = \{3, 4\}$ $\quad$ $E_3 = \{1, 4, 6\}$ etc.

If the outcome of the experiment is contained in E, then we say that E has occurred. Always $E \subseteq S$. Since E & S are sets, theorems of set theory may be effectively used to represent & solve probability problems which are more complicated.

**Example.** If by throwing a dice, the outcome is 3, then events $E_1$ and $E_2$ are said to hare occured.
In the child example – (i) If E, = {g}, then $E_1$ is the event that the child is a girl.
Similarly, if $E_2$ = {b}, then $E_2$ is the event that the child is a boy. These are examples of Simple events.

**Compound** events may consist of more than one outcome. Such as E = {1, 3, 5} for an experiment of throwing a dice. We say event E has happened if the dice comes up 1 or 3 or 5.

For any two events E and F of a sample space S, we define the new event $E \cup F$ to consists of all outcomes that are either in E or in F or in both E and F That is, the event $E \cup F$ will occur if either E or F or both occurs. For instances, in dice example (i) if event E = {1, 2} and F = {3, 4}, then $E \cup F = \{1, 2, 3, 4\}$.

That is $E \cup F$ would be another event consisting of 1 or 2 or 3 or 4. The event $E \cup F$ is called union of event E and the event F Similarly, for any two events E and F we may also define the new event $E \cap F$, called intersection of E and F to consists of all outcomes that are common to both E and F.

# WORKBOOK

**Example 1.** A box contains 5 white and 10 black balls. Eight of them are placed in another box. What is the probability that the latter box contains 2 white and 6 black balls?

**Solution.**

The number of balls is 15. The number of ways in which 8 balls can be drawn out of 15 is $^{15}C_8$. The number of ways of drawing 2 white balls = $^5C_2$. The number of ways of drawing 6 black balls = $^{10}C_6$

Total number of ways in which 2 white and 6 red balls can be drawn is $^5C_2 \times {}^{10}C_6$.

$\therefore$ The required probability $= \dfrac{^5C_2 \times {}^{10}C_6}{^{15}C_8} = \dfrac{140}{429}$

**Example 2.** Four cards are drawn at random from a peak of 52 playing cards. What is the probability of getting all the four cards of the same suit?

**Solution.**

For cards can be drawn from a deck of 52 cards in $^{52}C_4$ ways; there are four suits in a deck, each of 13 cards.

Thus, total number of ways of getting all four cards of same suit is

$$13C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 = 4\left({}^{13}C_4\right)$$

Hence, required probability

$$= \frac{4\left({}^{13}C_4\right)}{^{52}C_4} = \frac{198}{20825}$$

**Example 3.** The letters of word 'SOCIETY' are placed at random in a row. What is the probability that the three vowels come together?

**Solution.**

The letter in the word 'SOCIETY' can arranged in 7! Ways. The three vowels can be put together in 3! Ways. And considering these three vowels are one letter, we have 5 letter which can be arranged in 5! Ways.

Thus, favorable number of outcomes = 5! ×3!

Required probability $= \dfrac{5! \times 3!}{7!} = \dfrac{1}{7}$.

**Example 4.** In a race, the odds in favor of the four cars $C_1$, $C_2$, $C_3$, $C_4$ are 1:4, 1:5, 1:7, respectively. Find the probability that one of them wins the race assuming that a dead heat is not possible.

**Solution.**

The events are mutually exclusive because it is not possible for all the cars to cover the same distance at the same time. If $P_1$, $P_2$, $P_3$, $P_4$ are the probabilities of wining for the cars $C_1$, $C_2$, $C_3$, $C_4$, respectively, then

$$P_1 = \frac{1}{1+4} = \frac{1}{5} \qquad P_2 = \frac{1}{1+5} = \frac{1}{6}$$

$$P_3 = \frac{1}{1+6} = \frac{1}{7} \qquad P_4 = \frac{1}{1+7} = \frac{1}{8}$$

Hence, the chance that one of them wins

$= P_1 + P_2 + P_3 + P_4$

$= \dfrac{1}{5} + \dfrac{1}{6} + \dfrac{1}{7} + \dfrac{1}{8} = \dfrac{533}{840}$.

**Example 5.** Given $P(A) = \dfrac{1}{4}, P(B) = \dfrac{1}{3}$ and $P(A \cup B) = 1/2$, then what is the value of $P\left(\dfrac{A}{B}\right), P\left(\dfrac{B}{A}\right), P(A \cap B')$ and $P\left(\dfrac{A}{B'}\right)$?

**Solution.**

We know that

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{3} - P(A \cap B)$$

$$\Rightarrow P(A \cap B) = \frac{1}{12}$$

Thus, $P\left(\dfrac{A}{B}\right) = P\dfrac{(A \cap B)}{P(B)} = \dfrac{1/12}{1/3} = \dfrac{1}{4}$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/2}{1/3} = \frac{1}{4}$$

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# ESE OBJ QUESTIONS

**1.** A bag contains 7 red and 4 white balls. Two balls are drawn at random. What is the probability that both the balls are red?

**[ESE - 2017]**

(a) $\frac{28}{55}$ (b) $\frac{21}{55}$

(c) $\frac{7}{55}$ (d) $\frac{4}{55}$

**2.** A random variable X has the density function $f(x) = K\frac{1}{1+x^2}$, where $-\infty < x < \infty$. Then the value of K is

**[ESE - 2017]**

(a) $\pi$ (b) $\frac{1}{\pi}$

(c) $2\pi$ (d) $\frac{1}{2\pi}$

**3.** A random variable X has a probability density function

$$f(x) = \begin{cases} kx^n e^{-x}; & x \geq 0 \\ 0; & \text{otherwise} \end{cases} \text{ (n is an integer)}$$

with mean 3. The values of {k, n} are

**[ESE - 2017]**

(a) $\left\{\frac{1}{2}, 1\right\}$ (b) $\left\{\frac{1}{4}, 2\right\}$

(c) $\left\{\frac{1}{2}, 2\right\}$ (d) $\{1, 2\}$

**4.** What is the probability that at most 5 defective fuses will be found in a box of 200 fuses, if 2% of such fuses are defective?

**[ESE - 2017]**

(a) 0.82 (b) 0.79
(c) 0.59 (d) 0.82

**5.** 0If X is a normal variate with mean 30 and standard deviato 4, what is probability $(26 \leq X \leq 34)$, given A (z = 0.8) = 0.2881?

**[ESE - 2017]**

(a) 0.2881 (b) 0.5762
(c) 0.8181 (d) 0.1616